NAME: ________________________________

This exam is open-book and calculator. You may use any books or papers that you like. There are four problems in this exam, you have two full class periods. State your assumptions and reasoning for each problem. Justify your steps and clearly indicate your final answers.

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1. (25 points)

You are given the following two 1-D distributions which are valid for all values of $x$:

$$p(x|\omega_1) = \frac{1}{2}e^{-|x|}$$
$$p(x|\omega_2) = e^{-2|x|}$$

Assume that $P(\omega_1) = 1/3$ and $P(\omega_2) = 2/3$

(a) (5 points) Derive the Bayes classifier for this problem. In other words, how would you classify new data points $x$?
(b) (10 points) Sketch a graph that indicates the Bayes error. Compute the numerical value of the Bayes error for this problem.
(c) (10 points) Compute the value of the Bhattacharyya bound for this problem. Remember that these are not normal distributions.
2. (25 points) You are given two normal distributions with the following means and covariance matrices:

\[
\begin{align*}
\mu_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\mu_2 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\Sigma_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\Sigma_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}
\end{align*}
\]

Assume that \( P(\omega_1) = \frac{1}{4} \) and \( P(\omega_2) = \frac{3}{4} \)

(a) (5 points) Sketch one equal probability contour for each distribution. (Make sure that the contour you draw for each distribution specifies the same probability value).
(b) (10 points) Derive the analytic form for the Bayes decision boundary.
(c) (5 points) Sketch the Bayes decision boundary on a plot that also shows the equiprobability contours.

(d) (5 points) How would you classify the point \([0 \quad \frac{1}{2}]^T\)?
3. (25 points) Consider the following 1-D probability density function:

$$
\text{A-|} \\
| \ \ \\
| \ \ \\
| \ \ \\
|___
|---\--------
| \\
2/A
$$

Pardon the poor figure. It is a triangle hitting the y-axis at a value of \( A \) and hitting the x-axis at a value of \( 2/A \). It is zero everywhere else.

(a) (5 points) Write an equation for this probability distribution. Make sure that the distribution integrates to one.
(b) (20 points) Two points are sampled from this distribution—their values just so happen to be $x^{(1)} = 0$ and $x^{(2)} = 1$. What is the value of $A$ for the most likely distribution to have generated these two points?
4. (25 points) Short Answer.

(a) (5 points) Sample data comes from one class are given as:

\[
\begin{bmatrix}
0 & 1 \\
0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & -1 \\
-1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 \\
-1 \\
\end{bmatrix}
\begin{bmatrix}
-1 & 1 \\
-1 & 1 \\
\end{bmatrix}
\]

Compute the sampled mean and sampled covariance matrix. Make sure to use estimators that are unbiased.

(b) (5 points) You are given the heights and weights of a certain set of individuals in unknown units. Which one of the following six matrices is the most likely to be the sampled covariance matrix?

\[
\begin{array}{ccc}
1.232 & 0.013 & 1.232 & 0.867 & 1.232 & -0.867 \\
0.013 & 2.791 & -0.867 & 2.791 & -0.867 & 3.307 \\
1.232 & 3.307 & 1.232 & 0.867 & 1.232 & 3.307 \\
0.013 & 2.791 & 0.867 & 2.791 & 3.307 & 2.791 \\
\end{array}
\]
(c) (5 points) Find the vector $w$ which is specified by the Fisher criterion for problem 2.

(d) (5 points) After proper choice of $\omega_0$, do you expect the Fisher discriminant to perform better, worse, or the same as the discriminant derived in problem 2?
(e) (5 points) List all conditions under which the Bhattacharyya bound is exact for Normal distributions.